Analysis of Open-Type Waveguides by the Vector Finite-Element Method

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Abstract—**A novel variational expression that is suitable for the waveguide analysis is proposed. In addition, a new mapping technique to analyze open-type waveguides is introduced. Any arbitrarily shaped waveguides can be analyzed by the finite-element method with this variational expression and the mapping technique. The dispersion characteristics of the rectangular dielectric waveguide and the microstrip line are demonstrated and these results shows the validity and usefulness of this method well.**

I. Introduction

Numerically analyzing the variational expression by the finite-element method (FEM) is a powerful means to analyze the waveguide problems [1]. It can be applied to any arbitrarily shaped waveguides. But it has been difficult to apply the FEM to open-type waveguides.

In this letter, we propose a novel variational expression employing the electric field components as the trial functions. It is formulated in the form of a functional of propagation constant β. In addition, we introduce a new mapping technique to apply the FEM to open type waveguides. By using this technique, the variational expression is transformed into the algebraic eigenvalue problem, in which the propagation constant β is the eigenvalue. We can easily analyze open type waveguides by the previously mentioned method.

II. Variational Expression

The cross section of a waveguide is subdivided into a finite number of elements. The electric field $\boldsymbol{E}$ in each element is expressed as follows,

$$\boldsymbol{E} = (\boldsymbol{f}_t + j\beta \boldsymbol{i}_z g_z)\exp\{j(\mathrm{wt} - \beta z)\}, \qquad (1)$$

where $\boldsymbol{i}_z$ is the unit vector in the z direction, $\boldsymbol{f}_t$ and g_z are the transverse vector function and the scalar function, respectively. Substituting (1) into the Maxwell's equation, the following differential equations and the boundary conditions are derived:

$$\omega^2\epsilon\mu \boldsymbol{f}_t - \mu\nabla_t \times \frac{1}{\mu}\nabla_t \times \boldsymbol{f}_t - \beta^2(\boldsymbol{f}_t + \nabla_t g_z) = \boldsymbol{0}$$

$$\omega^2\epsilon\mu g_z + \mu\nabla_t \cdot \frac{1}{\mu}(\boldsymbol{f}_t + \nabla_t g_z) = 0. \qquad (2)$$

Manuscript received July 9, 1991.

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IEEE Log Number 9104367.

For the boundary between the element and the element:

$$\boldsymbol{i}_z \cdot (\boldsymbol{n} \times \boldsymbol{f}_t) = \text{continuous function}$$

$$g_z = \text{continuous function} \qquad (3)$$

$$\boldsymbol{n} \cdot \frac{1}{\mu}(\boldsymbol{f}_t + \nabla_t g_z) = \text{continuous function}$$

$$\boldsymbol{i}_z \cdot \frac{1}{\mu}\nabla_t \times \boldsymbol{f}_t = \text{continuous function}. \qquad (4)$$

For the boundary between the element and the electric wall:

$$\boldsymbol{i}_z \cdot (\boldsymbol{n} \times \boldsymbol{f}_t) = 0, \qquad g_z = 0. \qquad (5)$$

For the boundary between the element and the magnetic wall:

$$\boldsymbol{n} \cdot \frac{1}{\mu}(\boldsymbol{f}_t + \nabla_t g_z) = 0, \qquad \boldsymbol{i}_z \cdot \frac{1}{\mu}\nabla_t \times \boldsymbol{f}_t = 0, \qquad (6)$$

where $\boldsymbol{n}$ is a normal unit vector to the boundary of each element.

The following equation is the variational expression that satisfies (2) and the boundary condition:

$$\beta^2(\boldsymbol{f}_t, g_z) = \frac{\sum \int \frac{1}{\mu}(\omega^2\epsilon\mu |\boldsymbol{f}_t|^2 - |\nabla_t \times \boldsymbol{f}_t|^2)\, dS}{\sum \int \frac{1}{\mu}(|\boldsymbol{f}_t + \nabla_t g_z|^2 - \omega^2\epsilon\mu |g_z|^2)\, dS}, \qquad (7)$$

where $\sum$ represents a summation taken over all elements and $\int dS$ denotes the surface integration on each element. The trial functions $\boldsymbol{f}_t$ and g_z in (7) must satisfy the boundary conditions (4) and (5) for (7) to be the variational expression.

Equation (7) is a novel variational expression employing the electric field components as the trial functions. It is formulated in the form of a functional of the propagation constant β.

III. Application of the FEM to Open Type Waveguides

This section describes a mapping technique to apply the FEM to open type waveguides. First, we divide the cross section of a waveguide into two parts. One is the finite inner region S_1 including the main guiding structure, and the other is the infinite outer region S_2. Next, we introduce the coordinate transformation defined in (8) and let the infinite outer region S_2 be transformed into the finite region $\hat{S}_2$ shown in Fig. 1.

$$\hat{x} = \frac{x}{x^2 + y^2}, \quad \hat{y} = \frac{y}{x^2 + y^2}. \tag{8}$$

According to the procedure of the FEM, we divide the finite region S_1 and $\hat{S}_2$ into a finite number of triangular elements. The region S_1 including the complex guiding structure is not transformed, so S_1 can be divided easily. The region S_2 has the simple structure, so the transformed region $\hat{S}_2$ can be also divided easily.

The trial functions f_t and g_z in each element are expanded as follows,

$$f_t = \begin{cases} \sum_{m=1}^{6} N_m^{(1)}(x, y)\phi_m, & \text{in } S_1, \\ \sum_{m=1}^{6} \{\dot{D}_m(\hat{x}, \hat{y}) \cdot N_m^{(1)}(\hat{x}, \hat{y})\}\phi_m, & \text{in } \hat{S}_2, \end{cases}$$

$$g_z = \begin{cases} \sum_{m=1}^{6} N_m^{(2)}(x, y)\psi_m, & \text{in } S_1, \\ \sum_{m=1}^{6} N_m^{(2)}(\hat{x}, \hat{y})\psi_m, & \text{in } \hat{S}_2, \end{cases} \tag{9}$$

where ϕ_m and ψ_m are the unknown expansion coefficients. $N_m^{(1)}$ and $N_m^{(2)}$ are the linear vector shape function [1] and the quadratic scalar shape function [2], respectively.

$\dot{D}_m(\hat{x}, \hat{y})$ is the dyadic defined as follows:

$$\dot{D}_{2i-1} = \dot{D}_{2i} = \frac{1}{\hat{x}_i^2 + \hat{y}_i^2}\{(\hat{y}^2 - \hat{x}^2)(i_{\hat{x}}i_{\hat{x}} - i_{\hat{y}}i_{\hat{y}}) - 2\hat{x}\hat{y}(i_{\hat{x}}i_{\hat{y}} + i_{\hat{y}}i_{\hat{x}})\}, \tag{10}$$

where $(\hat{x}_i, \hat{y}_i)$ is the coordinate of a vertex i. The trial functions satisfying the boundary condition (4) and (5) are easily constructed by using the shape functions of (9).

Substituting (9) into (7) and using stationary condition of the variational expression, we can obtain a generalized eigenvalue problem as follows,

$$[A]\{\chi\} = \beta^2[B]\{\chi\}. \tag{11}$$

Here, $[A]$ and $[B]$ are real symmetric matrices describing the structure of the waveguide. $\{\chi\}$ is the column vector with the elements of ϕ_m and ψ_m. The propagation constant β of the guided mode can be obtained by solving (11).

IV. Numerical Examples

We first analyze the rectangular dielectric waveguide shown in Fig. 2. The width and the height of the core are a and b, respectively, ϵ_1, ϵ_2 and ϵ_0 are the permittivities of the core, the cladding and the vacuum, respectively. k_0 is the wavenumber in the vacuum.

Letting the boundary curve L between S_1 and S_2 be the circumscribed circle of the core, we divide S_1 and $\hat{S}_2$ into N triangular elements whose size is about the same. Using the symmetricity of the waveguide, a quarter of that is considered and let $N/4$ equal 104. Fig. 2 shows the dispersion characteristics and that is in good agreement with the results obtained by J. E. Goell [3] using the collocation method.

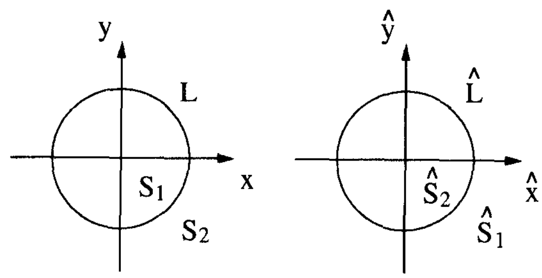

Fig. 1. Mapping from S_1, S_2 to $\hat{S}_1$, $\hat{S}_2$.

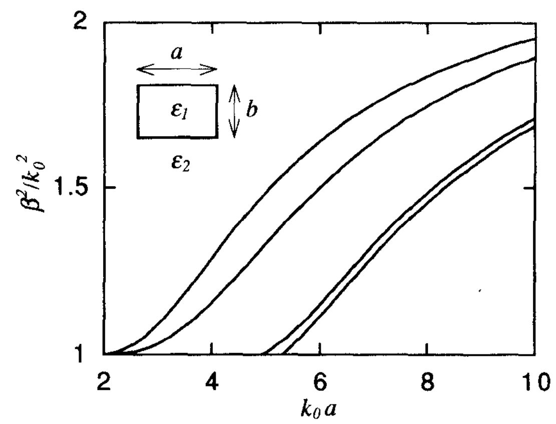

Fig. 2. Dispersion characteristics of the rectangular dielectric waveguide ($k_0 = \omega\sqrt{\epsilon_0\mu_0}$, $\epsilon_1 = 2.25\epsilon_0$, $\epsilon_2 = \epsilon_0$, $b = 0.5a$).

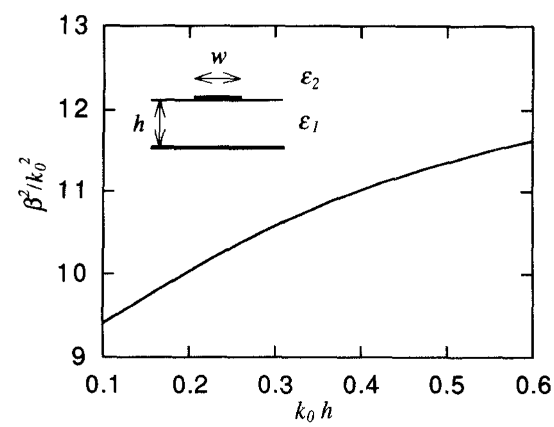

Fig. 3. Dispersion characteristics of the microstrip line ($k_0 = \omega\sqrt{\epsilon_0\mu_0}$, $\epsilon_1 = 13\epsilon_0$, $\epsilon_2 = \epsilon_0$, $w = 1.5h$).

The next numerical example has been carried out with the microstrip line shown in Fig. 3. The spacing between the strip and the ground conductor is h and the strip width is w. ϵ_1 and ϵ_2 are the permittivities of the dielectric substrate and the cladding, respectively. Fig. 3 shows the dispersion characteristics and that is in good agreement with the results obtained by X. Zhang *et al.* [4] using the FD-TD method.

V. Conclusion

In this letter, we propose a novel variational expression employing electric field components as the trial functions. It is formulated in the form of a functional of the propagation constant, so it suits waveguide analysis. In addition, we introduce a new mapping technique to apply the FEM to open type waveguides. Any arbitrarily shaped waveguide including open region can be analyzed easily by using the previously mentioned method.

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